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# PLANE FREE CONVECTION GENERATED BY A LOCAL HEAT SOURCE IN A STABLY STRATIFIED MEDIUM

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Solution of a linear system of Navier — Stokes equations is obtained in Boussinesq approximation for the plane motion of a viscous heat-conducting gas, induced by a local heat source in a stably stratified medium for a specified constant horizontal velocity of the oncoming stream. If the heat release is time independent, it is possible to separate in the case of stable stratification the asymptotically stationary flow region. The mechanism of the stationary flow formation is explained. A singularity of the flow related to the appearance downstream of the heat source of stationary waves whose length depends not only on the oncoming stream velocity but, also, on the vertical temperature gradient. It is shown that at some distance form the source the flow is unsteady and becomes divided into a number of vortices whose number depends on the convection development time and on the vertical temperature gradient.

In a stably stratified medium the vertical gradient of density inhibits vertical motion, and this introduces in the considered problem an inner scale determined by a characteristic dimension of the region in which the heat drainage induced by the upward motion and temperature gradient of the unperturbed medium takes place. Within that region the effect of viscosity on convection becomes significant when the Reynolds number determined by the inner scale is not too high. In such case the flow throughout the space can be defined by the system of equations of free convection.

1. Statement of the problem and its solution. Let a heat source of infinite length act in space in a direction normal to the plane $(x, z)$ (the $oz$ -axis is vertical). The source is defined by the quantity of heat $Q(x, z, t)$ released in a unit of time $t$ in the volume of unit area in the plane $(x, z)$ and of unit length in the direction of the $oy$ -axis. We consider heat sources for which

$$Q_0(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q\,dx\,dz \neq \infty \tag{1.1}$$

At some distance from the heat source a viscous heat conducting gas moves at constant velocity $u$ along the $ox$ -axis.

Let us consider the plane perturbed motion of gas induced by the indicated heat source. We define the motion of such gas by the system of Navier — Stokes equations in the Boussinesq approximation

$$\frac{\partial v_x}{\partial t} + u\frac{\partial v_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta v_x, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \tag{1.2}$$

$$\frac{\partial v_z}{\partial t} + u\frac{\partial v_z}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta v_z + g\frac{T'}{T_0}$$
$$\frac{\partial T'}{\partial t} + u\frac{\partial T'}{\partial x} + (\gamma_a - \gamma)v_z - \chi\Delta T' = \frac{Q}{\rho c_p}$$
$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0$$

where $u + v_x$ and $v_z$ are components of the perturbed stream velocity, $p$ is the medium dynamic pressure; $T' = T - T_0$, where $T$ is the temperature and $T_0(z)$ is the temperature of the unperturbed medium; $\rho$ is the density of gas; $\nu$ is the kinematic viscosity coefficient; $\chi$ is the thermal diffusivity coefficient; $\gamma_a = g / c_p$, where $g$ is the acceleration of gravity and $c_p$ is the specific heat capacity of gas at constant pressure; and $\gamma = -\partial T_0 / \partial z$.

We set $\nu = \chi = \text{const}$. To a stably stratified medium corresponds $\gamma = \text{const} < \gamma_a$.

We pass to the system of coordinates $(x'', z)$ moving along the $x$ -axis at constant velocity $u$ . In these coordinates the system of equtions of free convection is free of terms containing $u$ and the coordinates of the source are $Q(x + ut, z, t)$ (here and in what follows the primes at the variable $x$ are omitted). In the moving coordinate system the flow is of a local character and is dampened at infinity. This means that the flow has no potential part [1], and the system of equations of convection with boundary conditions reduces to the problem

$$\frac{\partial\Delta\psi}{\partial t} - \nu\Delta\Delta\psi = \frac{g}{T_0}\frac{\partial T'}{\partial x} \tag{1.3}$$
$$\frac{\partial T'}{\partial t} - \nu\Delta T' + (\gamma_a - \gamma)\frac{\partial\psi}{\partial x} = \frac{Q}{\rho c_p}$$
$$t = 0,\ \psi = 0,\ T' = 0;\ x = z = \pm\infty,\ \psi = 0, \qquad T' = 0$$

where $\psi$ is the stream function of the velocity field perturbation.

To solve the system of Eqs. (1.3) for $\psi$ we apply operator $\partial / \partial t - \nu\Delta$ to the first of these and operator $gT_0^{-1}\,\partial / \partial x$ to the second. Adding the obtained equations we obtain the problem

$$L\psi = g(\rho c_p T_0)^{-1}\partial Q / \partial x \tag{1.4}$$
$$t = 0, \quad \psi = 0, \quad x = z = \pm\infty,\ \psi = 0$$
$$L = \left(\frac{\partial}{\partial t} - \nu\Delta\right)\frac{\partial}{\partial t}\Delta - \nu\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta\Delta + \omega_0^2\frac{\partial^2}{\partial x^2}$$
$$\omega_0^2 = (\gamma_a - \gamma)\, g / T_0 = \text{const} > 0$$

whose solution we represent in the form

$$\psi = \frac{g}{\rho c_p T_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^t G(x, z, t \,|\, x', z', t')\frac{\partial}{\partial x'}Q(x' + ut', z', t')\,dt'\,dx'\,dz' \tag{1.5}$$

The problem for the Green's function $G$ is of the form

$$LG = \delta(x - x')\,\delta(z - z')\,\delta(t - t') \tag{1.6}$$
$$t = 0, \quad G = 0; \quad x = z = \pm\infty, \quad G = 0$$

To solve (1.6) we use the formal operator equation [2]

$$G = L^{-1}\delta(x - x')\,\delta(z - z')\,\delta(t - t') + G_0 \qquad (1.7)$$

which is equivalent to (1.6). In it $L^{-1}$ is the inverse of operator $L$ and $G_0$ is the nonsingular part of Green's function which is the solution of the homogeneous equation $LG_0 = 0$ and is determined with boundary conditions for function $G$ taken into account.

Representing $\delta$-functions in the form of Fourier expansion and taking into account (1.4), (1.5), and (1.7), we obtain

$$\psi = \frac{g}{\rho c_p T_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{t} Q(x' + ut', z', t')\,G^*(x, z, t \mid x', z', t')\,dt'\,dx'\,dz' \qquad (1.8)$$

$$G^* = -\frac{\partial G}{\partial x'} = -$$

$$\frac{1}{(2\pi)^3}\iiint \frac{ik_1 \exp\left[i\omega(t - t') + ik_1(x - x') + ik_2(z - z')\right]}{K(\omega, k_1, k_2)}\,d\omega\,dk_1\,dk_2$$

$$K = (k_1^2 + k_2^2)\left\{[i\omega + \nu(k_1^2 + k_2^2)]^2 + \omega_0^2 k_1^2 / (k_1^2 + k_2^2)\right\}$$

In deriving (1.8) we assumed that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{t} Q\,\frac{\partial G_0}{\partial x'}\,dx'\,dz'\,dt' = 0 \qquad (1.9)$$

It will be shown below that the assumption (1.9) follows from the condition of vanishing of the perturbation velocity components away from the heat source.

Integrating $G^*$ in (1.8) with respect to $\omega$, passing to the polar system of coordinates $(k, \varphi)$ using formulas $k_1 = k\cos\varphi$ and $k_2 = k\sin\varphi$, and integrating with respect to $k$, we obtain

$$G^* = -\frac{i}{8\omega_0[\pi^3\nu(t - t')]^{1/2}}\int_0^{2\pi}\Lambda[\zeta(\varphi)]\sin[\omega_0(t - t')\cos\varphi]\,d\varphi \qquad (1.10)$$

$$\Lambda(\zeta) = \exp(-\zeta^2)[1 - \operatorname{erf}(-i\zeta)]$$

$$\zeta(x, x', z, z', t, t'; \varphi) = \frac{(x - x')\cos\varphi + (z - z')\sin\varphi}{[4\nu(t - t')]^{1/2}}$$

Formula (1.10) together with (1.8) yields the solution of the problem. Passing to the fixed coordinate system, for the vertical velocity component we obtain

$$v_z = \frac{g}{8\pi^2\rho c_p T_0\omega_0\nu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{t}\frac{1}{t - t'}\,Q(x' + ut', z', t')\,I\,dt'\,dx'\,dz' \qquad (1.11)$$

$$I = \int_0^{2\pi}(1 + i\pi^{1/2}\zeta^*)\,\Lambda(\zeta^*)\cos\varphi\sin[\omega_0(t - t')\cos\varphi]\,d\varphi$$

$$\zeta^* \equiv \zeta(x, x' + ut, z, z', t, t'; \varphi)$$

2. The pattern of flow away from the heat source. We select the coordinate origin in the region in which $Q \neq 0$. In defining the flow at distances $R = (x^2 + z^2)^{1/2}$ that considerably exceed the characteristic dimension of the heat release region we substitute for $Q$ in (1.8) and (1.11) the expression $Q = Q_0(t')\,\delta(z') \times \delta(x' + ut')$ where $Q_0$ is determined by (1.1). We also assume that $R \gg (\nu t)^{1/2}$, $R \gg ut$. This means that we now consider the region which is virtually free of the heat from the source. There the flow is generated by the pressure field produced by the whole flow region, as well as by the perturbations of temperature $T'$, which are generated by the induced source $(\gamma_a - \gamma)v_z$. The effect of viscosity and thermal diffusivity in that region is negligibly small [3], hence it is possible to consider $\zeta^*$ to be large. The latter makes possible to write the formula for the stream function in conformity with (1.8) and (1.10) in the fixed coordinate system as

$$\psi = -uz + \frac{g}{2\pi\rho c_p T_0}\frac{x}{R^2}\int_0^t Q_0(t')\int_0^{t'} \cos\left[\Omega(t' - \tau)\right] J_0(\omega_0\tau)\,d\tau\,dt' \tag{2.1}$$

where $J_0$ is the Bessel function of the first kind of zero order and $\Omega = \omega_0 z / R$. It follows form (2.1) that convective motions are damped as $R \to \infty$. This confirms the applicability of condition (1.9).

If $Q_0$ is time independent, then with $t\Omega \gg 1$ from (2.1) we obtain for the problem the space-time asymptotic solution

$$\psi = -uz + (2\pi\rho c_p T_0 \omega_0^2)^{-1} g Q_0 \sin(\Omega t) / z \tag{2.2}$$

Note that formula (2.2) is invalid at the vertical axis $(x = 0)$, as well as near the horizontal axis where $\Omega \to 0$ and the inequality $t\Omega \gg 1$ cannot be satisfied.

The asymptotic formulas (2.1) and (2.2) show that at considerable distances from the source unsteady oscillations appear. There the quantities $\psi$, $v_z$, $v_x$, and $T'$ vary at each point of space with frequency $\Omega$ which depends on coordinates. The dependence on frequency $\Omega$ leads in time to the decomposition of flow into vortices whose number in every time interval $\tau = 2\pi / \omega_0$, increases by one in a single quadrant of the plane. The term vortex defines here a simply connected flow region within which vorticity is of the same sign. At the vortex boundaries $\psi = 0$. Each separate vortex borders on vortices with inverse vorticity sign. This property of flow is described in [4] in which the problem of convection development in a stably stratified atmosphere by local temperature perturbation was considered. The oscillatory character of flow is due to stable stratification $(\omega_0 > 0)$. In the case of neutral stratification $(\omega_0 = 0)$ oscillations are absent, and, as implied by (2.1), the solution is then of the form

$$\psi = -uz + (2\pi\rho c_p T_0)^{-1} g x R^{-2} \int_0^t t' Q_0(t')\,dt' \tag{2.3}$$

Comparison of (2.2) with (2.3) shows that the character of the velocity amplitude damping for both stable and neutral stratification as $R \to \infty$ is the same, i.e. $\vec{v} \sim R^{-2}$.

The asymptotic solutions (2.1) and (2.2) are also valid for nonlinear convection. However the conditions of their applicability are more rigid and dependent on the induced convection velocity.

3. The region of stationary flow. Let us consider the case when the source $Q = Q_0\delta(z')\delta(x' + ut')$ is linear and stationary. It follows from (1.11) that in the region $R \ll \sqrt{\nu t}$ when $u = 0$, $\omega_0 \neq 0$ and $\omega_0 t \to \infty$ a monotonic decrease of velocity takes place as the distance from the coordinate origin increases. This implies the existence of a solution of the plane problem which is stationary and damped at infinity. The possibility of appearance of a stationary flow region is related to the presence in the third of Eqs. (1.1) of the term $(\gamma_a - \gamma) v_z$ which can be considered as an induced heat drain. The characteristic dimension of the region in which the induced drainage of heat has a compensating effect on the source $Q$ is determined by the equality

$$(\gamma_a - \gamma) v_z{}^* l^2 = Q_0 / (\rho c_p) \qquad (3.1)$$

where the vertical velocity in the operation region of source $Q$ is to be taken as $v_z{}^*$ in (3.1). Setting $u^2 / (4\nu\omega_0) \ll 1$, from (1.11) we find that at point $x = z = 0$

$$v_z = (4\pi\rho c_p \nu)^{-1} [T_0(\gamma_a - \gamma)]^{-1/2} g^{1/2} Q_0 [1 - u^2 / (4\nu\omega_0)] \qquad (3.2)$$

which implies that the oncoming stream stabilizes convection.

From (3.1) and (3.2) we have

$$l = \{4\pi\nu / [\omega_0(1 - u^2 / (4\nu\omega_0))]\}^{1/2} \qquad (3.3)$$

Formulas (3.2) and (3.3) determine the applicability limit of the linear approximation for $R_1 = v_z l / \nu \ll 1$. The last condition in conjunction with (3.2) and (3.3) shows that the linear approximation is applicable for

$$Q_0 \ll (4\pi)^{1/2} \rho c_p \nu^{3/2} (\gamma_a - \gamma) / [1 - u^2 / (4\nu\omega_0)]^{1/2}$$

For arbitrary values of parameter $\beta = u / (2\sqrt{\nu\omega_0})$ the formula for the vertical velocity at the coordinate origin is in accordance with (1.11) of the form

$$v_z(\beta) = v_z(0)\, v_z{}^*(\beta)$$

$$v_z{}^*(\beta) = 1 - \frac{iu\beta}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \int_0^{2\pi} \Lambda(\tau) \cos^2\varphi \sin(t\cos\varphi)\, d\varphi\, dt$$

$$\tau = -\beta\sqrt{t}\cos\varphi$$

where $v_z(0)$ is determined by (3.2) with $\beta = 0$. Function $v_z{}^*(\beta)$ is satisfactorily approximated by the formula $v_z{}^*(\beta) = 2\pi^{-1} \operatorname{arctg}(\beta^{-2})$ which determines the character of the oncoming stream stabilizing effect on convection. As $\beta$ increases $v_z{}^*$ monotonically decreases from its maximum value equal unity to zero.

The interaction between the oncoming stream and the induced convection is similar to the stream flow at velocity $u$ on an obstacle which in this case is the convection zone perturbed by heat. Then with $u \neq 0$ a stationary flow independent of viscosity can be generated in the zone $l \ll R \ll \sqrt{\nu t}$. The characteristic dimension of the obstacle is $l$ and the "boundary layer" thickness $L$ in which viscosity is present is equal $(\nu l / u)^{1/2}$. This estimate is the more accurate the higher $R_2 = ul / \nu$. The pattern of flow at $R \gg L$ is obtained by setting in (1.8) and (1.10) $Q = Q_0\delta(z')\delta(x' + ut')$, indefinitely increasing $t$, and using ffor function $\operatorname{erf}(y)$ the asymptotic representation for considerable $y$. It follows then from (1.8) and

(1.10) that

$$\psi = -uz + \frac{gQ_0}{(2\pi)^2 \rho c_p T_0 \omega_0 u} \int_0^{\infty} \int_0^{2\pi} \frac{\sin(\omega_0 u^{-1} x' \cos\varphi)}{(x - x')\cos\varphi + z \sin\varphi} d\varphi \, dx'$$

The vertical velocity component at $z = 0$ is determined by formula

$$v_z(x) = \frac{gQ_0}{2\pi \rho c_p T_0 u^2} \int_0^{\infty} \frac{J_0(\omega_0 u^{-1} x')}{x - x'} dx' \qquad (3.4)$$

The integral in (3.4) is understood in the sense of principal value. For $x < 0$ and $\omega_0 u^{-1} \mid x \mid \gg 1$ from (3.4) we have

$$v_z(x) = (4\rho c_p T_0 u^2)^{-1} g Q_0 [N_0(\omega_0 u^{-1} \mid x \mid) - H_0(\omega_0 u^{-1} \mid x \mid)] \approx -(2\pi \rho c_p T_0 u \omega_0)^{-1} g Q_0 / \mid x \mid + O[(\omega_0 u^{-1} x)^{-3}] \qquad (3.5)$$

where $N_0$ and $H_0$ are, respectively, the Neumann and Struve functions. For positive $x$ from (3.4) we obtain

$$v_z(x) = (4\rho c_p T_0 u^2)^{-1} g Q_0 [N_0(\omega_0 u^{-1} x) + H_0(\omega_0 u^{-1} x)] \approx (2\sqrt{\pi} \rho c_p T_0 u \omega_0)^{-1} \sqrt{u / (\omega_0 x)} \sin(\omega_0 u^{-1} x - \pi/4) + O[(\omega_0 u^{-1} x)^{-1}] \qquad (3.6)$$

According to (3.5) and (3.6) the patterns of damping of the amplitude of vertical velocities ($v_z \sim \mid x \mid^{-1}$, $x < 0$) and ($v_z \sim x^{-1/2}$, $x > 0$) , respectively unstream and downstream is substantially different owing to the heat transfer by the oncoming stream to the region $x > 0$. The properties of convection upstream and downstream are qualitatively different. It follows from (3.6) that when $\omega_0 u^{-1} x \gg 1$ stationary waves of length $\lambda = 2\pi u / \omega_0$ are formed downstream of the heat source whose amplitude is dampened in accordance with the law $x^{-1/2}$ as $x \to \infty$ . There the gas flows downward thus compensating the upward motion (3.2) in the region of the heat source. Stationary waves are more pronounced in the first quadrant of the plane $(x, z)$.

We note in conclusion that for neutral classification ($\omega_0 = 0$) the linear solution (1.11) is divergent when $t \to \infty$. When $u = 0$ then at point $x = z = 0$ we have

$$v_z(t) = At / \nu, \quad A = gQ_0 / (8\pi \rho c_p T_0) \qquad (3.7)$$

Solution (3.7) is valid as long as $R_3 = v_z(t)\sqrt{t / \nu} \ll 1$. Condition $R_3 \ll 1$ imposes a restriction on time ($0 \leqslant t \ll A^{-2/3}\nu$) during which the linear solution (3.7) is valid. Although the oncoming stream (with $u \neq 0$) stabilizes convection, the linear approximation for $\omega_0 = 0$ also yields a divergent solution, namely, for $t \gg 4\nu / u^2$ at point $x = z = 0$ we have

$$v_z(t, u) = 2A \{1 - \pi^{-1} \ln [u^2 t / (4\nu)]\}$$

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